A PROPOSAL OF CATEGORISATION FOR ANALYSING INDUCTIVE REASONING

María C. Cañadas and Encarnación Castro

We present an analysis of the inductive reasoning of twelve Spanish secondary students in a mathematical problem-solving context. Students were interviewed while they worked on two different problems. Based on Polya's steps and Reid's stages for a process of inductive reasoning, we propose a more precise categorization for analyzing this kind of reasoning in our particular context. In this paper we present some results of a wider investigation (Cañadas, 2002).

Keywords: reasoning, inductive reasoning, stages, secondary students, conjectures.

Presentamos un análisis del razonamiento inductivo de doce estudiantes de educación secundaria en un contexto de resolución de problemas matemáticos. Los estudiantes fueron entrevistados mientras trabajaban en dos problemas diferentes. Basándonos en los pasos considerados por Pólya y Reid para un proceso de razonamiento inductivo, proponemos una categorización más precisa para analizar este tipo de razonamiento en nuestro contexto particular. En este documento presentamos algunos resultados de una investigación más amplia (Cañadas, 2002).

Palabras clave: razonamiento, razonamiento inductivo, pasos, estudiantes de secundaria, conjeturas.

Proof appears to be a real problem at different educational levels. On one hand, although Spanish pre-service teachers are accustomed to formal proof, they have difficulties in proof teaching (Cañadas, Nieto and Pizarro, 2001). On the other hand, secondary students do not make as much progress as they are supposed to in their reasoning. One possible reason lies in the fact that they cannot suddenly acquire the necessary reasoning skills for developing formal proof. They need a period of time to transform their daily reasoning into a formal one (Jones, 1996). Some studies show that primary and secondary students are able to formulate conjectures, examine and justify them if they start working from particular cases (Healy and Hoyles, 1998; Lampert, 1990). Some of these actions related to con-

Cañadas, M. C. & Castro, E. (2007). A proposal of categorisation for analysing inductive reasoning. *PNA*, *1*(2), 67-78.

jecturing are part of the inductive reasoning process. We are interested in analyzing the way secondary students go from particular cases to generalization.

This paper consists of four main parts. First, we present the theoretical framework of the study, which includes our proposal of categorization for inductive reasoning. Second, we outline the methodology of our empirical study. Third, we present some sample data organized according to the stages of our categorization and finally, we discuss some results of the study.

THEORETICAL FRAMEWORK

Inductive Reasoning

Inductive and deductive reasoning are the two traditional types of reasoning considered. We will focus on the first one although we are conscious that sometimes it is very difficult to separate them in students' work. Inductive reasoning is very important from a scientific viewpoint because it allows us to obtain scientific knowledge (Pólya, 1967). In the particular case of mathematics teaching, Pólya indicates that inductive reasoning is a method of discovering properties from phenomena and of finding regularities in a logical way. We will consider that inductive reasoning in Mathematics Education is a reasoning process that begins with particular cases and produces a generalization from these cases.

There is a confusing term that involves proof and reasoning: mathematical induction (MI). It is a formal method of proof based more on deductive than on inductive reasoning. Some processes of inductive reasoning conclude with MI but this does not always occur. For example, we cannot construct a correct proof by MI to justify that the angles in a triangle have a sum of 180º because the set of all triangles is not ordered.

From the curricular perspective, we find that at secondary level and in the two courses before the university, mathematics has as the main aim developing a certain level of reasoning and abstraction. At the end of these studies, students must master operations of abstract thinking that allow them to understand conjecture formulation, observation of particular cases, experimentation, hypothesis validation, and to elaborate explanations and theories structured in some way, etc. (Boletín Oficial del Estado, 2003a, 2003b). These actions are related to inductive reasoning, as we will see.

Stages in Inductive Reasoning

Pólya (1967) indicates four steps of a process of inductive reasoning: observation of particular cases, conjecture formulation based on previous particular cases, generalization and conjecture verification with new particular cases. In this context of empirical induction from a finite number of discrete cases, Reid (2002) describes the following stages: observation of a pattern, the conjecturing (with doubt) that this pattern applies generally, the testing of the conjecture, and the generalization of the conjecture. Based on these studies and taking into account our empirical work (Cañadas, 2002), we consider seven stages as describing the inductive reasoning process. In the following paragraphs, we explain these stages and we illustrate some of them in the context of the task of determining the maximum number of regions formed by *n* lines.

Observation of particular cases. The starting point is experiences with particular cases of the problem posed.

Organization of particular cases. The students' responses are different when they are able to organize particular cases in some way. They use different strategies to systematize and facilitate the work with particular cases.

Search and prediction of patterns. Observing particular cases (organized or not), we can think about the next, unknown case. In this sense, students are thinking about a possible pattern just for the cases they are observing. They are not thinking about applying the pattern to all cases.

Conjecture formulation. A conjecture is a statement based on empirical facts, which has not been validated. This "conjecture formulation" is like Reid's "conjecturing (with doubt)" which means making a statement about all possible cases, based on particular ones, but with an element of doubt. A clear example for this is when some students claim: "I think that you get the double of the number of straight lines" but they are not sure about that because they are thinking of what happens in the first two particular cases.

Conjecture validation. When students formulate a conjecture with doubt, they are convinced about the truth of their conjecture for those specific cases but not for other ones. At this stage, they try to validate their conjectures for new specific cases but not in general. In our example, they might validate their conjecture by drawing more than two lines.

Conjecture generalization. Mathematics patterns are related to a general rule, not only to some cases. Based on a conjecture which is true for some particular cases, and having validated such conjecture for new cases (conjecture validation), students might hypothesize that the conjecture is true in general.

General conjectures justification. The first step on the way to confirm or reject a general conjecture is validating it with particular cases. But this is not enough to justify a generalization. It is necessary to give reasons that explain the conjecture with the intention of convincing another person that the generalization is justified. At this point, a formal proof can provide the final justification that guarantees the veracity of the conjecture.

These stages can be thought of as levels from particular cases to the general case beyond the inductive reasoning process. Not all these levels necessarily occur, there are a lot of factors involved (as will be discussed below).

METHODOLOGY

The Problem Posed

Cañadas (2002) chose two tasks to analyze inductive reasoning in a group of twelve Spanish secondary students. Here we have chosen one of the tasks considered in that study to illustrate some of the results of the large set of data gathered. This task was proposed to the students in an interview context in the following way:

> *What is the maximum number of regions you can get in the plane if you draw straight lines?*

We notice here some specific characteristics of this problem that are considered for our analysis. The correct answer was unknown by the students so it is a problem for them. We attend that students recognize the general pattern. For this reason, we do not mention the number of lines they have to draw. Students must notice the functional relationship between the number of lines and the number of regions. This relationship relates to a second-degree polynomial, which appears in Spanish secondary curriculum.

Another important aspect was the way of representation. Although we posed the task in a verbal way, one of our interests is to analyze in which type of representations the students express their conjectures and their own justifications.

All the students started by drawing straight lines when they heard the problem. We show a correct (not the unique) way to solve this problem in Figure 1.

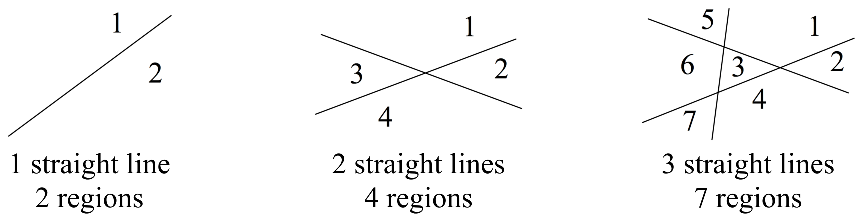

Figure 1. *First particular cases of this problem*

In Table 1 we organize the information concerning particular cases.

Table 1
First particular cases

Num. Straight lines	Num. Regions
1	2
2	4
3	7
4	11
5	16
…	…

With these data, we observe that with the first straight line, we get two regions; when we draw two lines, we get four regions (two more than in the previous case); when we draw three lines, we get seven regions (three more than in the previous case); when we draw four lines, we get eleven regions (four more that in the previous case), etc. Generalizing this pattern, we can say that when we draw the nth line, we get *n* new regions. In this sense, we can write the number of regions as $2+\sum_{i=2}^{n} i$. Developing this expression and calling a_n to the number of regions, we get: $a_n = 2+\sum_{i=2}^{n} i = 2+\left[\frac{n(n+1)}{2}-1\right] = \frac{n(n+1)}{2}+1$.

After students' conjecture formulation, they were asked to justify their conjectures. At secondary level, we did not expect for a formal proof. Our aim was to analyze if students can develop a particular way to justify their own conjectures and what characteristics these ways of justification have.

Interviews

We used individual interviews to observe the students' reasoning. The interviewer was one of the researchers and her aim was to propose the problem to the students and ask them questions in order that they explain what they were doing. She had an interview plan which allowed her to guide students though questioning so that we could observe their reasoning from particular cases to generalization.

Students

Secondary level is adequate to investigate what happens to inductive reasoning in this kind of problem from content and cognitive viewpoints. In primary level, mathematics is strongly connected to empirical facts and didactical materials. Spanish curriculum claims for secondary level that empirical-inductive reasoning must be reinforced in parallel to the use of deductive reasoning. In this sense, we observe that there is an evolution from the basic level of mathematics, based on empirical work and inductive processes, to high level mathematics, based mainly on mathematical structures and relationships among them.

We interviewed twelve Spanish students (six girls and six boys) from the four years of school before going to university (14-18 years old). We choose three students from each year with different academic results in order to obtain a wide variety of responses.

DATA COLLECTION AND ANALYSIS

We collected data in three complementary ways: the interviews were recorded on audio tape, we gave worksheets to the students so that they could write their work (if they wanted to) and the interviewer took notes during and after each interview about relevant details that could not be recorded on the tape.

We analyze these data in a qualitative way using Nud*ist revision (N4). This program allowed us to see the data in a structured way and to discover details, patterns and relations that would be more complicated to discover by hand.

For presenting and analyzing the data, we symbolize the students as 1, 2, 3 and 4 depending on the year to which they belong to. A, B or C indicates high, medium or low academic results. For example, 3A is a third year student whose academic results are higher than her/his classmates.

In our analysis, we notice that not all the stages in inductive reasoning are necessarily present in all tasks and not all students show the same stages for the same task. However, there are two relevant characteristics because they appear in all the stages:

Spontaneity. We analyze if students were able to advance in their reasoning by themselves or, on the contrary, they needed or even required the interviewer intervention.

Representation mode. We posed the problem in the verbal language. However, we consider four possible ways to express their reasoning related to this problem: verbal, arithmetic, geometric and algebraic.

Moreover, we noticed some general characteristics that facilitate the advance of one stage to the next one. These characteristics allow us to compare the work of these students in each stage but they can not be considered as sub-stages because there is no order relation among them. We summarize this in Table 2:

Table 2
Stages and characteristics

Stages	Characteristics
Observation of particular cases	Number of particular cases
	Type of particular cases
	Systematic way
Organization of particular cases	Tables
Search and prediction of patterns	Based on…
Conjecture formulation	Use of school knowledge
Conjecture validation	Based on…
Conjecture generalization	Characterization of even numbers
General conjecture justification	Justification necessity
	Based on particular cases
	General case

RESULTS

We will comment our results using the categorization described above and the related characteristics. We will summarize some of the main results in two tables. In Table 3 we present the responses related to the first two stages, which are related to particular cases. In Table 4 we will summarize results mainly related to conjectures.

Table 3
Particular cases

Student	Spontaneity	Number	Systematic	Organization
1C		4	×	
1B	×	5		×
1A	×	4		×
2C		3		×
2B		4		
2A	×	4	×	×
3C	×	5		×
3B	×	8	×	
3A		5		×
4C		4		

Observation of particular cases. As we can see in Table 3, seven students turned to particular cases in a spontaneous way, without any interviewer intervention. Finally (in the rest of the cases with the interviewer's suggestion), all of them considered a number of particular cases higher than three, so their work was similar in this respect. This common fact is relevant because they considered (some of them, explicitly) that the more particular cases they considered, the easier it would be to obtain a pattern. They had made a translation from verbal representation to the graphical one.

One difference in work with particular cases that had influence on the outcome of the problem solution was the systematic way of drawing the straight lines. 1C, 2A, 3B, 4B and 4A took into account the order of particular cases, starting by drawing one line and they continued with successive cases. In other cases, the students tried with different number of lines in no order. For example, 4C drew three lines and the next particular case he considered was drawing eight lines.

Organization of particular cases. Organizing particular cases can make it easier to observe a pattern. All of the students at this stage were able to express the particular cases in an arithmetic representation. As we can see in Table 3, seven of the students organized their particular cases in tables or equivalent ways like lists of numbers.

Search and prediction of patterns. The first pattern we observed in the students' reasoning was in graphical representation, when most of them decided that they had to draw the new line for a particular case, intersecting as many previous lines as they could.

We observed that it is very difficult to formulate a conjecture for students who have not organized the data.

Conjecture formulation. All of the students base their reasoning on the particular cases they have considered and as it is very difficult for them to obtain new particular cases, they refer to their knowledge about linear relationships. In this sense, we observed students who thought that they would get one more region than the number of lines they drew; students who thought that they would get twice as many regions; and students who tried to establish a proportional relationship. They were not convinced of these conjectures. They formulated them with some doubt because they could not be sure they were true in general.

Conjecture validation. In the previous stage, they simultaneously try to validate the conjectures with particular cases. It is the method they use to be more convinced about their own conjectures. Some students used particular cases to accept the conjecture. And for other students, these cases allowed them to notice that they had formulated an invalid conjecture. For example, when 1A thought that he would get twice as many regions, he claimed:

> *1A: yes. With two lines, four regions. Then with three lines, I must get six… but… no because I get seven.*

All of the students formulated a conjecture but in many cases, these conjectures were not correct. They observed whether their conjectures were true for the particular cases they considered before. The following reaction of these students' next was to verify their initial conjectures with all the particular cases they were able to draw. When they had a conjecture that was valid for them, they formulated a conjecture for the general case.

Table 4
Conjectures

Student	Infinite	Simple relationship	Linear			Recurrence	No recurrence generalization
			n+1	2n	Rule of 3		
1C		×		×			
1B		×		×			
1A		×	×	×	×		
2C	×	×		×	×	×	
2B	×	×					
2A	×	×	×			×	×
3C		×		×		×	

Table 4
Conjectures

Student	Infinite	Simple relationship	Linear			Recurrence	No recurrence generalization
			n+1	2n	Rule of 3		
3B	×			×		×	
3A	×	×		×			
4C	×	×		×	×	×	
4B	×			×		×	
4A	×	×		×		×	

Conjecture generalization. Eight students did not recognize the functional relationship between the number of straight lines and the number of regions at the beginning of their reasoning, although they had been working with some particular cases. These students claimed that, as they can draw as many lines as they want, they can get an infinite number of regions. The interviewer tried to guide them towards the objective of the problem.

The second general conjecture was considered by all the students except 3B and 4B. These ten students noticed that the more lines they draw, the more regions they get. This fact is indicated in Table 4 as "simple relationship".

All the students except 1A, 2B and 3A used algebraic language to express their conjecture for the general case. They clearly kept in mind the link between this kind of language and the generalization process.

Seven students noticed the recurrence relationship, observing the difference between two consecutive particular cases, although not all of them expressed it in the same way. Most of them, although they tried to use algebraic language, were able to express the recurrence in arithmetic or verbal language. In general, they had difficulties expressing the recurrence relationship in algebraic terms. For example, we observed student 3B, who feels completely confused in this sense:

> *3B: uff! [...] the difference between this case and the next one, we consider "z". Then "z" will be the difference and "x" is the number of lines. "y" is another number of regions and "a" is another number of regions. Then "z" will be... will be… ufff, I don't know.... […]. Let's see, here we have a set of regions, they will be "x", "y" and "h". Then the differences between "x" and "y" will be "z". The difference between "y" and "h" will be "z+1". The difference between "h" and "j" will be "z+2".*

Other students noticed that the recurrence relationship is useful on some occasions but not always. For example, 4B claims:

> *[...] I don't know how to express numbers... there is something here that not... I don't know how to write that because if I use the number of lines*

that we had in the previous case, then I need to know the other number... and that number is what we had in the previous too plus…

But although some students recognize the limitations of the recurrence relationship, just one of them (2A) detected a different way to generalize the functional relationship.

General conjecture justification. No student recognized the necessity of justifying their results on their own. They saw the result as an evident consequence from particular cases, without needing any additional justification to be convinced of its truth.

DISCUSSION

We show a categorization of seven stages for describing the inductive reasoning process. These have arisen from concrete tasks but they have a general character that permits us to use them for other problems. Every stage admits some characteristics that depend on a number of factors, such as the kind of task or the students involved. We have presented them in a general way, so they might be applied for other problems, too.

Inductive reasoning appeared implicitly or explicitly in the work of all the secondary students interviewed. Students turned to particular cases when they tried to get a general pattern, so we can conclude that inductive reasoning appears naturally at these educational levels. These students show a tendency to take an empirical approach rather than to work with mathematical structure. We are now collecting new Spanish data that suggests the same idea, which is in contrast with the curricular expectations of students at the end of secondary level.

The seven students who justified the general conjecture found a mathematical pattern from particular cases obtained from the characterization used in their justifications. This confirms that searching for patterns is a relevant and necessary step in inductive reasoning process in lower university levels and this kind of work can provide students a way to get the generalization process in a more significant form. In this sense, if students do not have habits of mind needed in order to discover a mathematical structure from particular cases, it will be very difficult for them to work with mathematical structures significantly in later courses.

Many students considered the conjecture obvious on the basis of particular cases and did not think that a general conjecture justification was necessary to validate their statements. Spanish students at these levels are still not accustomed to prove their conjectures. In some way, students have validated their conjectures with some particular cases and some of them recognize the need to prove the general expression to be convinced of its truth. This kind of activity can be used to introduce students to justification tasks.

We have found two main points where students have difficulties. The first point was in finding the pattern. Even when they organized some particular cases of the underlying pattern, they did not know how to obtain the pattern. They considered the linear pattern but did not try the quadratic function as a possible pattern, in spite of their work with it in mathematics classes. The other difficult point was when they had found the pattern –the recurrence relation in most cases– and they did not know how to express the pattern.

Algebraic language appeared in all cases when the students tried to express the generality of the pattern. So generalization activities can be considered as a way to introduce algebra, as Mason (1996) pointed out.

In general, students do not feel sure with their own work and they need — even require— the interviewer's intervention.

We did not notice significant differences among students' reasoning in the different courses of secondary level. It happened in the same way with students with different academic results belonging to the same year. We only detected some differences in the way they expressed their argumentations.

Acknowledgements

We wish to thank David Reid for his remarks on a previous version of this paper.

REFERENCES

Cañadas, M. C. (2002). *Razonamiento inductivo puesto de manifiesto por alumnos de secundaria*. Granada: Universidad de Granada.

Cañadas, M. C., Nieto, E., & Pizarro, A. (2001). El valor de la demostración en la educación secundaria. In J. Berenguer, B. Cobo, & J. M. Navas (Eds.). *Actas de investigación en el aula de matemáticas: retos de la educación matemática del siglo XXI* (pp. 167-172). Granada: Universidad de Granada.

Healy, L., & Hoyles, C. (1998). *Justifying and proving in school mathematics. (TechnicalrReport on The Nationwide Survey)*. London: Institute of Education, University of London.

Jones, L. (1996, July). *A developmental approach to proof.* Paper presented at the 8th International Congress on Mathematical Education, Seville.

Küchemann, D., & Hoyles, C. (2002). Students' understanding of a logical implication and its converse. In A. Cockburn, & E. Nardi (Eds.). *Proceedings of the 26th International Group for the Psychology of Mathematics Education, Norwich UK*, Vol. III. (pp. 241-248). Norwich, UK: University of East Anglia.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27*, 29-36.

Boletín Oficial del Estado (2003a). *Real Decreto 831/2003 de 27 de junio, de Ordenación General y las Enseñanzas Comunes de la Educación Secundaria Obligatoria* (Vol. BOE Nº 158, pp. 25683-25743). Madrid: Ministerio de Educación y Ciencia.

Boletín Oficial del Estado (2003b). *Real Decreto 832/2003 de 27 de junio, de Ordenación General y las Enseñanzas Comunes del Bachillerato* (Vol. BOE Nº 159, pp. 26039-26099). Madrid: Ministerio de Educación y Ciencia.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra. Perspectives for research and teaching* (pp. 65-86). London: Kluwer.

Pólya, G. (1967). *La découverte des mathématiques*. París: DUNOD.

Reid, D. (2002). Conjectures and refutations in grade 5 mathematics. *Journal for Research in Mathematics Education, 33*(1), 5-29.

This document was originally presented at IV CERME, Sant Feliu de Guixols (Girona), Spain, 2005.

María Consuelo Cañadas
Universidad de Zaragoza
mconsu@unizar.es

Encarnación Castro
Universidad de Granada
encastro@ugr.es