SPONTANEOUS META-ARITHMETIC AS THE FIRST STEP TOWARD SCHOOL ALGEBRA

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Taking as a point of departure the vision of school algebra as a formalized meta-discourse of arithmetic, we have been following six pairs of 7th-grade students (12-13 years old) as they gradually modify their spontaneous meta-arithmetic toward the "official" algebraic form of talk. In this paper we take a look at the very beginning of this process. Preliminary analyses of data have shown, unsurprisingly, that while reflecting on arithmetic processes and relations, the uninitiated 7th graders were employing colloquial means, which could not protect them against occasional ambiguities. More unexpectedly, this spontaneous meta-arithmetic, although not supported by any previous algebraic schooling, displayed some algebra-like features, not to be normally found in everyday discourses.

Keywords: Algebra; Discourse; Formalization; Generalization; Meta-arithmetic

La meta-aritmética espontánea como el primer paso hacia el álgebra escolar

Tomando como punto de partida la visión del álgebra escolar como un meta-discurso formalizado de la aritmética, hemos estado siguiendo a seis pares de estudiantes de 7° curso (12-13 años) cuando modifican gradualmente su meta-aritmética espontánea hacia la forma algebraica "oficial" de hablar. En este artículo miramos el principio de este proceso. Los análisis preliminares de los datos han mostrado, como era de esperar, que mientras reflexionaban sobre los procesos y relaciones aritméticas, los alumnos no iniciados emplearon medios coloquiales que no evitaban las ambigüedades ocasionales. Más inesperadamente, esta meta-aritmética espontánea, a pesar de no apoyarse en ninguna enseñanza algebraica previa, desplegó algunas características similares al álgebra que no se encuentran normalmente en los discursos diarios.

Términos clave: Álgebra; Discurso; Formalización; Generalización; Meta-aritmética

The idea that algebra is a language—e.g., of science—has been with us for centuries, and so was the controversy over this description (Lee, 1996). In our attempts to follow the development of school children's algebraic thinking we take as a point of departure a definition that responds to some of the concerns voiced by the objectors of the algebra-as-language approach. We define algebra as a discourse, that is, a form of communication. This approach, while preserving the centrality of the motif of language, transfers algebra from the category of passive tools to that of human activities. This ontological change has important ramifications for how we view the development of algebraic thinking and how we investigate it. This paper is a report on the initial phase of our ongoing study of this topic. In this project, we have been following algebraic discourse of six pairs of 7th graders from its beginnings in the form of spontaneous talk on numerical processes and relations, and through the subsequent process of its gradual formalization in school.

SCHOOL ALGEBRA AS FORMALIZED META-ARITHMETIC

The definition of algebra as a discourse is a derivative of our foundational assumption that thinking is an individualized form of interpersonal communication (Sfard, 2008). To communicate either with others or with oneself, one has to act according to certain rules, implicitly shared by all the interlocutors. Different types of tasks and situations may evoke different sets of communicational regulations, that is, different discourses. Algebra can be defined as a sub-category of mathematical discourse that people employ while reflecting on arithmetical relations and processes.

Let us take a closer look at the two basic types of meta-arithmetical tasks that give rise to algebra. First, there is a question of numerical patterns, which we describe formally with the help of equalities, such as, say, $a \cdot (b + c) = a \cdot b + a \cdot c$. Although nothing in this latter proposition says so explicitly, this is, in fact, a piece of meta-arithmetic. Indeed, the symbolic proposition $a \cdot (b + c) = b \cdot a + c \cdot a$ is a shortcut for the sentence: "To multiply a number by a sum of other two numbers, you may first multiply each of the other two numbers by the first one and then add the results." This type of meta-arithmetic narrative can be called generalization. The other algebra-generating tasks are questions about unknown quantities involved in completed numerical processes. This type of task is described in the modern algebraic language as solving equations. Indeed, equations, say $2x + 1 = 13$, are meta-questions on numerical processes. In the present case the question is: "What number, if doubled and increased by 1, would yield 13?"

According to this definition, algebraic thinking begins when one starts scrutinizing numerical relations and processes in the search for generalization or in an attempt to solve equations. The narratives—propositions about

mathematical objects—that result from these two types of activities do not have to employ any symbolic means. Here is a rather striking historical example of pre-symbolic algebra taken from the Indian text known as *Aryabhatiya* (499 AD):

> *Multiply the sum of the progression by eight times the common difference, add the square of the difference between twice the first term, and the common difference, take the square root of this, subtract twice the first term, divide by the common difference, add one, divide by two. The result will be the number of terms.* (Boyer & Mertzbach, 1989, p. 211)

Although hard to recognize, this lengthy piece presents a solution of an equation: it is a prescription for finding a number of elements in an arithmetic progression, whose first term, the difference and the sum are given. While considering the communicational shortcomings of this intricate rendering it is easy to understand why formalization of the discourse was one of the major trends in the further development of algebra. Formalization was a process that aimed at increasing the effectiveness of meta-arithmetic communication. This goal required three types of action: (a) *disambiguation*, that is prevention of the possibility of differing interpretations of the same expressions by different interlocutors; (b) *standardization*, supposed to ensure that all the interlocutors follow the same communicational rules; and (c) *compression*, which turns lengthy statement such as the one quoted above into concise, easily manipulable expressions. This latter goal may be attained in the twin action of reification and symbolization. Reification means turning narratives about processes into ones about objects (cf. the notion of nominalization in Halliday & Martin, 1993). Reifying usually involves introduction of nouns—e.g., sum or product—with which to replace lengthy verb clauses. The above quote from *Aryabhatiya*, although formulated as a description of a process—a sequence of numerical operations: note the verbs multiply, add, etc—, includes compound noun clauses, such as "the square of the difference between twice the first term, and the common difference", which reify sub-sequences of computational steps. Symbolization means replacement of nouns, predicates, and verbs with ideograms, that is, symbols referring to objects the way words do, but without being uniquely tied to specific sounds. To make the replacement possible, a change in the grammar of the propositions may sometimes be necessary. For example, when a purely processual verbal description is translated into standard symbolic expression, the order of appearance of arithmetic operations may no longer correspond to the order of their implementation. When presented in the canonic symbolic manner, *Aryabhatiya*'s rule reincarnates into the concise expression $(\sqrt{8Sd + (2a - d)^2} - 2a + d)/2d$, the special property of which is that it can be used both as a prescription for a calculation and as a result of this calculation.

METHOD OF STUDY

In this section we describe the methodological details regarding the empirical study like the goal, participants, procedure and analysis.

Goal

The overall goal of our study is to contribute to the project of mapping the development of algebraic thinking in school. If algebra is a formalized meta-arithmetic, child's algebraic discourse may be expected to emerge from discourses that the child has already mastered and which she can now try to adjust to the meta-arithmetical tasks of finding numerical patterns and investigating computational processes. In our study, therefore, the learning of algebra has been conceptualized as a gradual closing of the gap between students' spontaneous meta-arithmetic and the formal algebraic discourse to which they are exposed in school. The aim of our investigations is to describe this process in as detailed a way as is feasible and useful.

Participants and Procedure

Considering our goal, six pairs of Hebrew-speaking 12-13 year old 7th-grade Israeli students have been interviewed at intervals of about two months—each pair is noted by a pair of letters: H-T, A-S, etc—. Each round of interviewing consists of five to six meetings lasting for 60 to 90 minutes. The first round began just before the students were introduced to algebra in school. At the time this paper is being written, this first round has been completed, the interviews transcribed and partially analyzed, and the second round of interviewing is about to begin. We intend to conduct four rounds altogether, with the last one commencing about 18 months after the first.

Tools

In each round, the interviewees are asked to complete a battery of tasks that can be organized in the three-dimensional matrix, with the following binary distinctions constituting the three dimensions:

- ◆ Informally (InF) versus Formally (For). The task is stated informally, thus encouraging spontaneous meta-arithmetical talk; or formally, for example by using canonic algebraic symbolism, thus inviting formal algebraic solution.
- ◆ Generalization (Gen) versus Equation (Equ). The task invites a generalization or solving an equation.
- ◆ Real-life (ReL) versus Abstract (Abs). The task is set in real-life or abstract context.

Each of the resulting eight categories can be subdivided even further. For example, in the case of equations, we included tasks with numerical data (Num) and also tasks that ask for a parametric (Par) solution—this locates this latter

type of task in the mixed genre of equation-solving and generalization—. Table 1 presents two samples of the tasks prepared for the first, pre-algebraic round of interviews.

Table 1
Sample Tasks from the First Round of Interviewing

Type	Statement
	Task 1
<InF, Gen, Abs>	(a) Given the sequence: 4, 7, 10, 13, 16…. Write the next three elements of the sequence.
	(b) What number appears in the 20th place in the sequence?
	(c) What number appears in the 50th place in the sequence?
	(d) Write a rule for calculating the number that appears in any place in the sequence.
	Task 2
<For, Equ, ReL>	On the shelf, there are books in English and in Hebrew. The number of English books exceeds the number of Hebrew ones by 8. If there are n books altogether, how would you calculate the number of those in English? (Par)

Note. InF = informally; Gen = generalization; Abs = abstract; For = formally; Equ = equation; ReL = real-life; Par = parametric solution.

Analysis

To map the development of the discourse, we describe and then compare samples of students' meta-arithmetic discourse collected in the successive rounds of interviewing. The descriptions focus on four defining characteristics of the discourse: (a) its *keywords*—e.g., those that denote variables or unknowns—and their use; (b) its *visual mediators*—icons, algebraic ideograms, graphs—, and their use; (c) its *routines,* that is, patterned, recurrent forms of discursive actions; and (d) *narratives* that the interviewees endorse and label as true. The specific questions that guide our examination of some of these discursive features will be presented in the next section, along with our sample responses.

SOME FINDINGS FROM THE FIRST ROUND OF INTERVIEWING

Our study has only begun, but even so, in this brief paper we can only present a small fraction of our findings so far. We restrict this report to the description of the students' spontaneous activity of generalization, as observed when they tried

to solve Task 1. The activity of generalization has been investigated by many researchers (Lannin, Barker, & Townsend, 2006; Radford, Bardini, & Sabena, 2007; Zazkis & Liljedahl, 2002). In most of these studies the principal focus was on strategies—routines—students used to detect and describe patterns. In the present study, while employing techniques of discourse analysis, we distribute our attention evenly between all four characteristics of the discourse produced in the activity of generalizing: the use of words, visual mediators, routines and endorsed narratives. Due to the scarcity of space, we restrict our present account to the first two of them. While reporting, we list only the most salient of the observed phenomena—the salience is not formally assessed; the evaluation of relative frequencies of these phenomena is yet to be completed—.

Words and Their Use
The analysis of the use of words focuses on (a) the choice of verbal tools for generalizing and (b) the syntax.

The Choice of Verbal Tools for Generalizing
The first question that guides our analysis of meta-arithmetic regards the verbal means students use to generalize, that is, to perform the necessary saming. This last term, saming, regards the linguistic change that is the very essence of the process of generalizing: replacing specific numbers (e.g., 3, 5, 7, …) with a single signifier—odd number—so as to turn infinitely many similarly structured arithmetic expressions—e.g., the square of 3, the square of 5, etc.—into a single meta-arithmetic expression—the square of an odd number—. The saming signifier is called *variable*, which in formalized algebra usually comes in the form of a Latin letter, such as *x* or *y*. In the beginning of our study, the participants have not yet been introduced to algebraic symbolism, and it is thus not surprising that when asked in Task 1, Part d, to write "a rule for calculating the number that appears in any place in the sequence", they used familiar words as their saming devices. This way of dealing with saming is instantiated several times in the following rule, written by one of the participants, H—non-italic are noun clauses, each of which does the job of saming over a specific set of numbers—:

> *To find* a certain place in the sequence *I need* the place that I found—*it better be round—and then 3—or any other number that is the* regularity—*times* what must be added to the number you have now and then to add the number you have now *and the product of the* regularity *and* what you still need*, and that's it.*[4] (pair T-H)

This rule was written after the girls calculated the 50th element of the sequence—Part c of task—by adding $30 \cdot 3$ to the 20th element, found

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[4] This transcription and the rest of this paper have been translated from Hebrew by the authors.

previously, in response to Part b of Task 1. Translated into formal symbolic language, it would yield the following formula: $a_n = a_m + (n - m) \cdot 3$, or even $a_n = a_m + (n - m) \cdot d$, where $a_n, n, n - m, a_{n-m}$, and d are the traditional symbols for elements of arithmetic progression. In particular, here are the formal translations of all noun clauses from the quote:

a certain place in the sequence *is equivalent to* a_n

the place that I found *seems to appear here in the double role of*

– a_m *[note that H speaks about adding a multiple of 3 to this number]* *and of*

– m *[note the parenthetic remark "it better be round"]*

regularity *is equivalent to d [the common difference of arithmetic progression]*

what must be added to the number you have now —*contrary to what the words seem to be saying*—*the use [it is said to be multiplied by 3] indicates that it is* $n - m$ *that is meant here*

the number you have now is equivalent to a_m

This single example alerts us to a number of important phenomena, which we observed on many occasions, in the talk of this participant as well as of many other students. The phenomena are deemed important because they probably need to be considered in planning the further process of formalization of the students' spontaneous meta-arithmetic.

As might be expected, the student opted for generalizing words that hinted at their prospective roles in the problem. In the single proposition quoted above, the hinting is done in several ways. Some of the words function as metaphors, as is the case when the index of an element is called a *place*. Some other names are metonymies, that is, represent the whole by its part. This is the case when the nth element, a_n is called the place I found. Interestingly, there is also the "reverse" of metonymy. In the expression "what must be added to the number you have now", which is meant to signify $n - m$, the whole—what must be added—appears in the role of its own part: of a multiplier with the help of which the addend is to be produced. Finally, the student made use of the genus, that is, of a broader category to which the given object belongs. This is the case when the common difference of the given progression, 3, is called *regularity* and when the girl refers to an element of the sequence as number—admittedly, this latter word was suggested by the designers of the task—. To overcome overgeneralizations, H uses specifying descriptions, such as "the number you have now." The term regularity has not been restricted by an additional description and this fact has two ramifications: First, the student has achieved a higher level of generalization

than required by the authors of the task—the authors asked for a computational rule for the sequence in which the specific number, 3, must be added in the transition from any element to its successor—. In a sense, therefore, she did even better than expected. Second, however, there is no hint in the generalizing word regularity that the regularities considered in the problem are those that produce arithmetic progressions. The resulting rule, therefore, is not self-explanatory and may even be dismissed by some interpreters as offering only a special case of what it promises to present.

The Use of Words (Syntax)

The main question asked with respect to the syntax of the participants' spontaneously composed generalizing propositions regards the degree of reification: Do the propositions speak about doing—calculations—or about properties of objects? Indeed, reifying is the key move toward disambiguation and condensation of meta-arithmetic narratives and may thus be seen as a "signature" feature of formal algebraic sentences.

The formerly discussed lengthy proposition from *Aryabhatiya*, although processual in its general tone, contained noun clauses that reified several of its sub-processes. It is striking that a similar partial reification appeared in our young participants' spontaneous meta-arithmetic sentences. Note, for example, the H's clause "the product of the regularity and what you still need", that speaks about a result—product—of an operation—multiplication—rather than about the operation as such. This property is even more salient in another version of the rule for calculating any element of the given arithmetic sequence, which the same student, H, produced toward the end of the session: "the place times the regularity of the sequence plus one" (pair T-H).

This time, the "rule" does not even sound as a prescription for action: It does not contain any verbs—times and plus are not verbs!—and does not constitute a full sentence. Unlike in the case of the previous version, no structural change would be necessary to translate it into the canonic symbolic formula $n \cdot d + 1$.

Visual Mediators and Their Use

The salient property of our participants' meta-arithmetic was the scarcity of visual mediation other than arithmetical—numerical—expressions. Those of the students, who did try to express their rules with the help of ideograms, used either letters or markers such as boxes or lines. Thus, for example, the two students whose work was discussed above presented the simplified version of their rule as $\square \cdot 3 + 1$ (pair T-H). It should be stressed that in most cases, the students' interpretation of boxes was different from that of letters: Whereas letters functioned mainly as names of objects, the box was usually understood as a marker of a physical space for numbers. Indeed, unlike in the case of letters, which were supposed to signify the same number in all their appearances,

identically looking boxes—squares—were often used indiscriminately for all the variables in the problem.

Thus, in our study, some of the students presented rules such as this one in the form $\square \cdot 3 + 1 = \square$. Interestingly, one of the participants wrote $x \cdot 3 + 1 = x$ (pair A-S), the use clearly inspired by his former experience with squares functioning as delineators of a physical space for numbers.

DISCUSSION: WHERE THE STUDENTS ARE AND WHAT COMES NEXT

With an eye to the ultimate goal of informing instruction, we focused our efforts on identifying dissimilarities between students' spontaneous meta-arithmetic and the formal algebra taught in schools. Let us stress that the discussion that follows and the tentative answers given in the end are grounded in a body of data much richer and more extensive than could be presented in this brief paper.

Colloquial, spontaneously developed discourses are known for their occasional blurriness and vagueness. Therefore, it did not come to us as surprise that upon close examination, the texts produced by our participants, although quite impressive in their resourcefulness, proved also full of ambiguities. Consider H's complex prescription for calculating any element of the arithmetic progression. Here, H used a single noun for a number of purposes—see her metaphoric use of the word place for the index of an element and the metonymic use of the same term for the element itself—and, on another occasion, referred to a single object in a number of ways—e.g., note the difference between the expressions "the place I found" and "the number you have now," both of which were used with reference to the previously calculated element—. She also used generic names which were all too general and, as such, could be easily misinterpreted by her interlocutors. To overcome overgeneralizations, H employed specifying descriptions, such as "the number you have now." However, this type of specification, being context-dependent—note the use of the deictic words *you* and *now*—could not possibly bar multiple interpretations.

All this said, our study, so far, has resulted also in some less predictable findings. On the basis of our own previous research (Sfard & Linchevski, 1994), we conjectured that the students' spontaneous meta-arithmetic would be about processes rather than objects. It is because of this prediction that we were careful to formulate the first tasks in processual language. For example, in Task 1, Part d we asked for the rule for calculating any element of the sequence rather than inquiring about what such generic element is. We were thus quite surprised to find out remarkable structural similarities between the students' verbal meta-arithmetic and the formal reified algebra. Two possible explanations come to mind when we try to account for this finding. First, structures of algebraic formulas are not unlike those of arithmetic expressions, and thus our students

might just be building on their knowledge of the latter type of structure. Second, it is possible that these days algebra is simply "in the air": elements of algebraic discourse may be present in other school discourses well before its formal introduction in the 7th-grade. With the help of media, algebraic forms of expression may even be infiltrating colloquial discourses. To check these conjectures, we decided to broaden our study and to conduct similar interviews with 6th and 5th-grade students.

Whatever the results of these latter investigations, we believe that one of the present tentative conclusions from our study is unlikely to change: While much work must be invested in formalization of students' spontaneous meta-arithmetic, the resources with which children are coming to their algebra classrooms may be a much better foundation for the development of formal algebraic discourse than could be expected on the basis of what is known about their mathematical education so far. The more knowledgeable we are about these resources, the better our chances for helping the students in closing the gap between their spontaneous meta-arithmetic and the formal algebra taught in school. Above all, we need this knowledge to be able to teach in such a way as to preserve the all-important link between the two discourses.

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